Synchronization in large directed networks of coupled phase oscillators

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We d he emergence of collec i e nchroni a ion in large direc ed ne ork of he erogeneo o cilla or b generali ing he cla ical K ramo o model of globall co pled pha e o cilla or o more reali ic ne ork . We e end recen heore ical appro ima ion de cribing he ran i ion o nchroni a ion in large ndirec ed ne ork of co pled pha e o cilla or o he ca e of direc ed ne ork . We al o con ider he ca e of ne ork i h mi ed po i i e-nega i e co pling reng h . We compare o r heor i h n merical im la ion and nd good agreemen . © *2005 American Institute of Physics*. (DOI: 10.1063/1.2148388)

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I. INTRODUCTION

The cla ical K ramo o model[13,14] de cribe a collec ion of globall co pled pha e o cilla or ha e hibi a ran i ion from incoherence o nchroni a ion a he co pling reng h i increa ed pa a cri ical al e. Since real orld ne ork picall ha e a more comple r c re han all- o-all co pling,[15,16] i i na ral o a k ha effec in erac ion r c- re ha on he nchroni a ion ran i ion. In Ref. 12, e died he K ramo o model allo ing general connec i i of he node , and fo nd ha for a large cla of ne ork here i ill a ran i ion o global nchron a he co pling reng h e ceed a cri ical al e k_c. We fo nd ha he cri ical co pling reng h depend on he large eigen al e of he adjacenc ma ri A de cribing he ne ork connec i i . We al o de eloped e eral appro ima ion de cribing he beha ior of an order parame er mea ring he coherence pa he ran i ion. Thi pa ork a re ric ed o he ca e in hich $A_{nm}=A_{mn}$ 0, ha i , ndirec ed ne ork in hich he co pling end o red ce he pha e difference of he o cilla or .

Mo ne ork con idered in applica ion are direc ed,[15,16] hich implie an a mme ric adjacenc ma ri , $A_{nm} \neq A_{mn}$. Al o, in ome ca e he co pling be een o cilla or migh dri e hem o be o of pha e, hich can be repre en ed b allo ing he co pling erm be een he e o cilla or o be nega i e, A_{nm} 0. The effec ha he pre ence of direc ed and mi ed po i i e-nega i e connec ion can ha e on nchroni a ion i , herefore, of in ere . Here e ho ho o r pre io heor can be generali ed o acco n for he e o fac or . We d e ample in hich ei her he a mme r of he adjacenc ma ri or he effec of he nega- i e connec ion are par ic larl e ere and compare o r he- ore ical appro ima ion i h n merical ol ion .

Thi paper i organi ed a follo . In Sec. II e re ie he re l of Ref. 12 for ndirec ed ne ork i h po i i e

man he erogeneo co pled pha e o cilla or . Thi i a ion
can be modeled b he eq a ion

$$\dot{}_{n} = {}_{n} + k \sum_{m=1}^{N} \square$$

A eraging o er he freq encie , one ob ain he *frequency distribution approximation* (FDA)

$$r_n = k \sum_m A_{nm} r_m \int_{-1}^{1} g(zkr_m)\sqrt{1-z^2}\,dz. \qquad (13)$$

The al e of he cri ical co pling reng h can be obained from he freq enc di rib ion appro ima ion b le - ing $r_n \to 0^+$, prod cing

$$r_n^{(0)} = \frac{k}{k_0} \sum_m A_{nm} r_m^{(0)}, \qquad (14)$$

here $k_0 = 2/\pi g(0)$. The cri ical co pling reng h h corre pond o

$$k_c = \frac{k_0}{\lambda}, \qquad (15)$$

here i he large eigen al e of he adjacenc ma ri A and $r^{(0)}$ i propor ional o he corre ponding eigen ec or of A. B con idering per rba ion from he cri ical al e a $r_n = r_n^{(0)} + \delta r_n$, e panding $g(zkr_m)$ in Eq. (13) o econd order for mall arg men , m l ipl ing Eq. (13) b $r_n^{(0)}$ and m- ming o er n, e ob ained an e pre ion for he order parame e er pa he ran i ion alid for ne ork i h rela i el homogeneo degree di rib ion [17]

$$r^2 = \left(\frac{1}{k_0^2}\right)\left(\frac{k}{k_c} - 1\right)\left(\frac{k}{k_c}\right)^{-3}, \qquad (16)$$

for $0 < (k/k_c) - 1 \ll 1$, he2r4115.078hj/F6.768598406.2813Tm2764444111..9.9789j/F599Tc-307.9h37859.di rib ion 6.913Tm2764444

$$r = \square_{n=1}^{N} r$$

the adjacency matrix is independently chosen to be 1 with probability s and 0 with probability $1-s$, and the diagonal elements are set to zero.) Even though the network constructed in this way is directed, for most nodes $d_n^{\rm in} \approx d_n^{\rm out}$. For $N=1500$ and $s=2/15$, Fig. 1(a) shows the average of the order parameter r^2 obtained from numerical solution of Eq. (1) averaged over ten realizations of the network and frequencies (triangles), the frequency distribution approximation (FDA, solid line), and the mean field theory (MFT, long dashed line) as a function of k/k_c, where the results for the FDA and the MFT are averaged over the ten network realizations (note, however, that the FDA and the MFT do not depend on the frequency realizations). (The perturbation theory Eq. (16) agreed with the frequency distribution approximation and was left out for clarity.) The error bars correspond to one standard deviation of the sample of ten realizations. We note that the larger error bars occur after the transition. When the values of the order parameter are averaged over ten realizations of the network and the frequencies, the results show very good agreement with the frequency distribution approximation and the directed mean field theory.

In order to study how well our theory describes single realizations, we show in Fig. 1(b) the order parameter r^2 obtained from numerical solution of Eq. (1) for a particular realization of the network and frequencies (boxes), the time averaged theory (short dashed line), and the frequency distribution approximation (solid line) as a function of k/k_c. As can be observed from the figure, in contrast with the time averaged theory, the frequency distribution approximation deviates from the numerical solution (boxes) by a small but noticeable amount. This behavior is observed for the other realizations as well. We note that the FDA and MFT results are virtually identical for all ten realizations. On the other hand, the TAT and the results from direct numerical solution of Eq. (1) show dependence on the realization. Since the FDA and MFT incorporate the realization of the connections A_{nm}, but not the frequencies, we interpret the observed realization dependence of the TAT and the direct solution of Eq. (1) as indicating that the latter dependence is due primarily to fluctuations in the realization of the frequencies rather than to fluctuations in the realization of A_{nm}.

Note that for our example $N=1500$ and $s=2/15$ implies that on average we have $d^{\rm in} \approx d^{\rm out} \approx 200$. Thus for comparison purposes, we generated an undirected network as follows: Starting with=a Eq.a9F54825ek-2 with TDfrem realiza 4direc re , a in he ndirec ed ca e, he al e of he a erage of he order parame er ob ained from n merical ol ion of Eq. 1 . The direc ed per rba ion heor gi e a good appro i-ma ion for mall al e of k clo e o k_c, a e pec ed. On he o her hand, he direc ed mean eld heor predic a ran i-ion poin hich i maller han he one ac all ob er ed.

When n mericall ol ing Eq. 32 b i era ion of Eq. 33, on ome occa ion a period o orbi a fo nd in ead of he de ired ed poin . If e deno e he lef hand ide of Eq. 33 b z_n^{j+1} and he righ hand ide b $f(z_n^j)$, e fo nd ha con ergence o a ed poin a facili a ed b replacing he righ hand ide b $(z_n^j+f(z_n^j))/2$ and nding he ed poin of hi modi ed em.

In hi e ample, a lo co pling reng h ro ghl $k/k_c \lesssim 4$, here k_c i comp ed from Eq. 37 he order parame er comp ed from n merical ol ion of Eq. 1 i maller han ha ob ained from he TAT and FDA. A k increa e , ho e er, he TAT and FDA heorie cap re he a mp o ic al e of he order parame er r. We no e ha in hi ca e he a mp o ic al e i larger han ha corre ponding o pha e locking i.e., he one ob ained b e ing $_n=0$ in Eq. 35, r 0.54 0.46=0.08, hich e indica e b a hori on al do -da hed line in Fig. 4, and m ch maller han $r=1$, he al e corre ponding o no fr ra ion i.e., $_n$ $_m$ =0 for A_{nm} 0 and for A_{nm} 0 in Eq. 35. The mall cale of he hori on al a i i d e o he fac ha e are plo ing r^2, and o o r de ni ion of he order parame er hich a ign a al e of 1 o a nonfr ra ed con g ra ion. The mall al e of he order parame er indica e a rong fr ra ion.

We no e ha in hi e ample, in con ra i h he e ample di c ed o far, here i aria ion in he al e of he order parame er predic ed b he FDA for differen reali a ion of he ne ork. Thi indica e ha , a he e pec ed al e of he co pling reng h A_{nm} become mall i.e., q 1/2 mall, c a ion d e o he reali a ion of he ne ork become no iceable. Al ho gh he al e predic ed b he FDA and TAT depend on he reali a ion of he ne ork and freq encie , e no e for k/k_c 6 ha he e al e rack he al e ob er ed for he n merical im la ion of he corre ponding reali a ion. A an ill ra ion of hi , e plo in Fig. 5 he al e of r^2 ob ained from he TAT ar and he al e of r^2 ob ained from he FDA diamond er he al e ob ained from n merical ol ion of Eq. 1 for k/k_c =8. Each poin corre pond o a gi en reali a ion of he ne ork, i h re l a eraged o er en reali a ion of he freq encie . The ellip e rro nding he ar TAT da a ha e er ical and hori on al half- id h corre ponding o he andard de ia ion of r^2 TAT and r^2 im la ion for he en freq enc reali a ion . The half- id h of he hori on al bar on he diamond FDA da a indica e he andard de ia ion of r^2 im la ion lations in networks with a much larger number of connections per node, and the effect of fluctuations would likely be reduced.

We also considered a case in which the adjacency matrix is asymmetric and has mixed positive-negative connections. For $N=1500$ nodes, we constructed an adjacency matrix by setting its nondiagonal entries to 1, −1, and 0 with probabilities 8/45, 4/45, and 11/15, respectively. The latter probabilities yield an expected number of connections of 400. Our theories work satisfactorily in this case, and, since the results are similar to those in Fig. 3, we do not show them. In this case there is no guarantee that there is a real eigenvalue (as needed for estimating the critical coupling strength in Eq. (15)), or that the largest real eigenvalue (if there is one) has the largest real part. Numerically, we find that for matrices constructed as in this example there is a real positive eigenvalue and that, furthermore, it is well separated from the largest real part of the remaining eigenvalues (see Fig. 6). We also find this for other values of q provided $|q-\frac{1}{2}|$ is not too small. We provide a discussion of this issue and how the of he non ero en rie being cho en randoml e.g., in he mme ric ca e, he po i ion of he non ero en rie i cho en hen con r c ing he ne ork , ing he con g ra ion model, and heir al e being al o de ermined randoml from a gi en probabili di rib ion e.g., 1 i h probabili q and 1 i h probabili 1 q. O r in ere i foc ed on he gap be een he large real eigen al e if here i one and he large real par of he o her eigen al e . In Ref. 23 he pec r m of cer ain large par e ma rice i h a erage eigen al e 0 and ro m $_{m=1}A_{nm}=1$ a de cribed and a he ri ic anal ical approach a propo ed. U ing re l for ma rice i h ero mean Ga ian random en rie ,[24] Ref. 23 predic ha he pec r m of he non-Ga ian random ma-rice he con ider con i of a ri ial eigen al e =1 i h he remaining eigen al e di rib ed niforml in a circle cen ered a he origin of he comple plane i h radi

$$=\sqrt{N}\ ,\qquad (A1)$$

here ^{2} i he ariance of he en rie of he ma ri . We nd ha hi approach al o cceed in de cribing he pec r m of he ma rice in o r e ample . In o r ca e, he diagonal en rie are 0, o ha he a erage eigen al e i al o 0 a in Ref. 23. We nd ha here i al a a large real eigen al e appro ima el gi en b he mean eld al e

$$=\langle\tilde{d}^2\rangle/\langle\tilde{d}\rangle\qquad (A2)$$

ee Ref . 12 and 25, here $\tilde{d}_n=\sum_{m=1}^{N}A_{nm}$ and $\langle\tilde{d}^2\rangle=\sum_{n=1}^{N}\tilde{d}_n^2$, hich in he ca e con idered in Ref. 23 red ce o =1. We al o n mericall con rm ha he remaining eigen al e are niforml di rib ed in a circle of radi a de cribed in Ref. 23. Thi i ill ra ed in Fig. 6.

Th for N if here i a gap of i e be-een he large real eigen al e and real par of he re of he eigen al e pec r m. U ing Eq . A1 and A2 i can be ho n ha , for ne ork i h large eno gh n mber of con-nec ion per node or i h eno gh po i i e or nega i e bia in he co pling reng h, here i a ide epara ion be een he large eigen al e and he large real par of he remain-ing eigen ec or . For mme ric ma rice , imilar re l ap-pl i.e., he b lk of he pec r m of he ma ri A can be appro ima el ob ained a de cribed abo e ing Wigner' emicircle la .

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