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Matching in Auctions with an Uninformed Seller*

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Abstract

In many auctions, matching between the bidder and seller raises the value of the contract for both parties although information about the matching may be incomplete. We consider the case in which each bidder observes the quality of his match with the seller but the seller does not observe the quality of the matches. Our objective is to determine whether it is in the seller's interest to (1) account for matching in his allocation decision and (2) observe the matches prior to the auction.

It is shown that irrespective of how important matching may be to the seller, the optimal mechanism can be implemented without using matching as a factor. If the seller has commitment power, he can raise his expected utility further by observing the matches ex ante. However, if the seller cannot commit, his value for the information may be negative: the seller's knowledge of the matches generates an asymmetry across bidders which depresses bids. The more matching matters, the greater the penalty associated with observing the matches in advance.

1 Introduction

The real value of a contract lies beyond its financial components: the degree to which the

not the matches of his opponents with the seller). In contrast, the seller does not observe his matches with the bidders.

The reader will note that the notion of matching advanced here is different from the notion advanced in the two-sided matching literature.[1] While the latter use the term to refer to the pairing of agents in a two-sided market, our paper uses the term to refer to the compatibility between a bidder and seller. Since this compatibility induces a positive correlation between the valuations of the bidder and seller, our notion of matching is more closely related to the literatures on affiliated values (e.g., Milgrom and Weber, 1982a) and interdependent valuations (e.g., Jehiel and Moldovanu, 2001), but while these literatures are more concerned with linking the bidders' valuations, our paper focuses on linking the valuations of the bidder and seller.

We solve for the optimal mechanism and find that it can be implemented via a standard first-price auction with an appropriate choice of reserve price. Since a better match implies a higher contract value, well matched bidders face a higher opportunity cost of not raising their bids. As a result, bids increase in match. By awarding the contract to the highest bidder, the seller finds himself automatically contracting with the best matched bidder. Since a first-price auction does not (directly) account for matching, we conclude that the seller need not consider matching as a factor in his allocation decision – no matter how important matching may be to the seller.

[1] See Gale and Shapley (1962), Shapley and Shubik (1972), Crawford and Knoer (1981), Kelso and Crawford (1982), Kamecke (1998), Hatfield and Milgrom (2005), and Bulow and Levin (forthcoming).

Implementing the first-price auction may require commitment power. Since the seller's utility is determined by both price and match, the seller may prefer a lower price and higher match to a higher price and lower match. Hence, if after observing the bids the seller believes the winning bidder is a poor match, he may be inclined to deviate from the "high bid wins" rule and offer the contract to another bidder instead.

The question naturally arises: can the optimal mechanism be implemented without commitment? In order to answer this question, we examine a *first-score* auction, in which each bidder bids on price alone but the seller selects the bidder whose combination of price and expected match maximizes his expected utility.[2] Since this allocation rule reflects the seller's true preferences, there should be no incentive for the seller to deviate from the rule ex post.

We find that the equilibrium bidding strategies in the first-score auction are identical to those in the first-price auction. In a first-score auction, well matched bidders have an incentive to convey their information to the seller in order to raise their probability of winning. Since well matched bidders have a higher value for the contract, they can credibly signal their favorable matches by raising their bids beyond the point at which it is profitable for poorly matched bidders to mimic them.[3] Given that higher bids signal better matches,

[2] Che (1993), Branco (1997), Zheng (2000), and Asker and Cantillon (2004) analyze a similar auction format in which the winning bidder is selected on the basis of price and quality. Our mechanism differs in that bidders bid only on price and the seller is left to estimate match on his own. Our mechanism is more closely linked to the biased procurement problem studied by Rezende (2004), in which bidders bid only on price but the allocation rule incorporates a bias determined by the seller. Our environment differs in that the bias is linked to the bidder's private information about his match.

[3] Bikhchandani and Huang (1989), Katzman and Rhodes-Kropf (2002), Das Varma (2003), Goeree (2003), Haile (2003), and Molnár and Virág (2004) examine signaling in auctions, but these papers are concerned with bidders signaling their private information to other bidders so as to affect future strategic interactions. In contrast, the signaling behavior in our paper is motivated by the structure of the auction game itself: bidders are interested in signaling their private information to the seller in order to influence the seller's choice of winner. In this sense, our paper is more similar to Avery (1998), which addresses the use of jump bids to signal a high valuation and encourage competing bidders to withdraw.

We then introduce an opportunity for the seller to observe the matches before the auction. Under commitment, the value of the information is positive since the seller can extract the entire surplus by making a take-it-or-leave-it o®er to the best-matched bidder. However, in the absence of commitment, the seller is unable to appropriate all the rent because he cannot reject o®ers which exceed his reservation value but are una®ordable for any other bidder.

Moreover, when the seller lacks commitment power, his knowledge of the matches may depress bids. If the seller has observed the matches prior to the submission of bids, a well matched bidder knows that the auction is biased in his favor irrespective of the o®er he makes. As a result, he need not bid as aggressively to win. In fact, the bias permits a well matched bidder to bid less than a poorly matched counterpart and still win the auction. Thus, there is an incentive for well matched bidders to capitalize on their advantage by reducing their bids. We call this e®ect the *asymmetry e®ect*.[5]

Since the asymmetry e®ect reduces bids, the value of the information under no commitment is not only lower than it is under commitment but may actually be negative. The greater the e®ect of matching on the seller's utility, the greater the advantage enjoyed by a well matched bidder and the greater the incentive to reduce his bid. Therefore, we ¯nd

[5] There is a vast literature on the negative e®ect of asymmetries on price competition, the majority of

that the more the seller cares about matching, the stronger his incentive not to observe the matches in advance.

This result lies in stark contrast to the conventional wisdom that bidders derive their pro¯ts from their private information (see Milgrom (1981), Milgrom and Weber (1982a), Milgrom and Weber (1982b), and McAfee and McMillan (1987)). The di®erence arises from the two modi¯cations made to the standard independent private values model: relaxing the assumption that the seller has commitment power and augmenting the seller's utility function to account for matching. In this framework, well matched bidders can only gain from (veri¯ably) disclosing their private information: the lack of commitment power prevents the seller from driving the price above the competitive level, and the bias in favor of well matched bidders dampens price competition even further.[6]

The remainder of this paper is organized as follows. Section 2 introduces the model. In Section 3, we solve for the optimal mechanism and identify which allocation rules implement the optimal mechanism in the commitment and no commitment case, respectively. We also derive a set of conditions under which the optimal outcome can be achieved in the absence of commitment. Section 4 examines the extent to which obtaining information about the matches ex ante is of value to the seller. Special attention is paid to the no commitment case and the role of the asymmetry e®ect in reducing bids. Conclusions are o®ered in Section 5.

2 The Model

A seller is to auction off a contract to one of n risk-neutral bidders ($n \geq 2$).[7] Every potential pairing of seller and bidder has an associated match. We denote the match between the seller and bidder i by $\mu_i \in [\underline{\mu}, \overline{\mu}] \subset \mathbb{R}$, where

Assumption 2 *$V : \left[\underline{\theta}, \overline{\theta}\right] \to \mathbb{R}$ is continuous and strictly increasing over $\left[\underline{\theta}, \overline{\theta}\right]$.*

The seller's utility is zero if he does not contract with any bidder.[8]

The following assumption is imposed so as not to rule out the possibility of a mutually beneficial trade:

Assumption 3 (participation condition) *$V\left(\overline{\theta}\right) + \overline{\theta}$ is positive.*

Additionally, we impose the following regularity condition:

Assumption 4 (regularity condition) *The function*

$$V(\theta_i) + \theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)}$$

is strictly increasing over $\left[\underline{\theta}, \overline{\theta}\right]$.[9]

We assume bidder i is better informed about his match than the seller is. Bidder i observes his match θ_i but not the matches of his opponents.[10] The seller does not observe the matches directly, and therefore, his beliefs about the matches are determined by the prior, F, and the observed bids.

3 The Optimal Mechanism

Intuition suggests that if matching affects the seller's expected utility, the seller should account for matching in his allocation decision. In fact, we observe this behavior in a number

[8] When $V(\theta_i) = v\theta_i$, where v is a positive constant, our model can be mapped to the interdependentvieg0TD[

of environments. For example, PDVSA accounted for technological complementarities in its

Similarly, the seller's expected utility from the mechanism is

$$U_0 \equiv E_\theta \left[\sum_{i=1}^{n} V(\theta_i)\, p_i(\theta) + \sum_{i=1}^{n} t_i(\theta) \right]. \tag{3.2}$$

The mechanism is optimal if it maximizes U_0 subject to

Incentive compatibility (IC): $\quad U_i(\theta_i; \theta_i) \geq U_i(x; \theta_i) \quad \forall i; \forall \theta_i; \forall x;$

Individual rationality (IR): $\quad U_i(\theta_i; \theta_i) \geq 0 \quad \forall i; \forall \theta_i;$

and

$p_i(\theta) \geq 0$ and $\sum_{i=1}^{n} p_i(\theta) \leq 1 \quad \forall i; \forall \theta.$

Proposition 1 *The optimal mechanism satisfies*

$$p_i(\theta) = \begin{cases} 1 & \text{if } \theta_i \geq \theta_j \ \forall j \text{ and } \theta_i \geq \theta^* \\ 0 & \text{otherwise} \end{cases}$$

and

$$E_{\theta_{-i}}[t_i(\theta_i; \theta_{-i})] = \begin{cases} \theta_i F^{n-1}(\theta_i) - \int_{\theta_*}^{\theta_i} F^{n-1}(x)dx & \text{if } \theta_i \geq \theta^* \\ 0 & \text{otherwise} \end{cases}$$

where

$$\theta^* = \begin{cases} \left\{ x \in \left[\underline{\theta}; \bar{\theta}\right] : V(x) + x - \frac{1 - F(x)}{f(x)} = 0 \right\} & \text{if } V(\underline{\theta}) + \underline{\theta} - \frac{1}{f(\underline{\theta})} < 0 \\ \underline{\theta} & \text{otherwise.} \end{cases}$$

Proof: See Appendix. $\Box$

The proof develops a relaxed optimization program by reducing the number of choice variables from two to one. It then identifies the unique solution of the relaxed program, and demonstrates that it satisfies the constraints of the original program. The regularity condition plays a key role.

Note that the mechanism outlined in Proposition 1 can be implemented via a standard first-price sealed-bid auction with an appropriate choice of reserve price. Since a higher match implies that the bidder has a higher value for the contract, well matched bidders can afford to bid more than poorly matched bidders. Hence, a rule that allocates the contract to the highest bidder also allocates the contract to the bidder with the best match.

Since the allocation rule in a first-price auction does not (directly) incorporate matching, we conclude that the seller can maximize his expected utility without using matching as a factor in his allocation decision – no matter how important matching may be to the seller.

However, implementing a first-price auction may require commitment power. Since $V(\theta^*) + \theta^* > 0$, there exists a mutually beneficial trade between the seller and a bidder with type θ^*. As a result, we encounter the usual problem associated with an elevated reserve price: when the high bid falls just short of θ^*, the seller may prefer awarding the contract to retaining it.

The fact that the seller's utility is a function of both price and match generates an additional problem. In the absence of commitment, the seller is not bound by the "first price wins" rule announced at the auction's outset. Consequently, if the seller believes the quality of the winning bidder's match is sufficiently poor, he may elect to award the contract to another bidder, whose price offer is lower but whose match is believed to be higher. Hence, if the seller's inability to commit is common knowledge, bidders select their strategies assuming the contract goes to the bidder whose combination of price and expected match maximizes the seller's expected utility.

In the following section, we address these issues in investigating whether the optimal mechanism can be implemented without commitment.

3.2 Implementation under No Commitment

Consider the following auction game:

1. Each bidder submits a bid independently and simultaneously.

2. The seller contracts with the bidder whose offer maximizes the seller's expected utility provided that the offer is not less than the seller's reservation utility of zero. That is, bidder i wins the contract if

$$E[V(\theta_i) \mid b_i] + b_i \geq 0$$

and

$$E[V(\theta_i) \mid b_i] + b_i > E[V(\theta_j) \mid b_j] + b_j \quad \forall j \neq i.$$

Ties are resolved by a random draw with equal probability.

3. Once the contract is allocated to bidder i, θ_i is revealed. The seller's payoff is $V(\theta_i) + b_i$, bidder i's payoff is $\theta_i - b_i$, and all other bidders get zero. The auction game is then over.

We call this auction game a *first-score* auction, where the term "score" refers to the combination of bid and expected match. For instance, bidder i's score is given by $E[V(\theta_i) \mid b_i] + b_i$. As indicated in the timeline above, the contract is allocated to the bidder with the highest score provided that that score is nonnegative.

The first-score auction is much like a first-price auction in that the winning bidder pays his bid. However, unlike a first-price auction, the winner is the bidder who offers the most attractive combination of bid and expected match. This allocation rule allows the seller to reject an offer made by the highest bidder and allocate the contract to another bidder whose pairing of bid and expected match is more attractive than the pairing offered by the highest bidder. Moreover, the first-score auction does not require an elevated reserve price: the seller retains the contract only when every offer falls short of the seller's reservation utility. Since the allocation rule of the first-score auction reflects the seller's true preferences, there is no incentive for the seller to deviate from it after observing the bids.

Our use of a first-score auction to represent the seller's lack of commitment is consistent with the literature. Che (1993) states that in the absence of commitment "the only feasible scoring rule is one that reflects the seller's [true] preference ordering," and Rezende (2004) allows a seller without commitment power to renege on the announced allocation rule and select the auction winner arbitrarily. Our representation is also consistent with the principal-agent model in Bester and Strausz (2000): they define imperfect commitment in terms of a two-stage game, in which agents select messages in the first stage and the principal updates his beliefs and selects an allocation in the second stage.

Our approach will be to investigate the perfect Bayesian equilibria of the first-score auction to see whether the optimal mechanism outlined in Proposition 1 can be implemented in the absence of commitment.

Let $P_i(b)$ represent bidder i's probability of winning with a bid of b. Since the seller observes only the bids offered and not the vector of types, each bidder's probability of winning the auction is a function of his bid but not of his type. Let $B_i : \mathbb{R} \times [\underline{\mu}; \overline{\mu}] \to [0; 1]$

represent bidder i's equilibrium bidding strategy (possibly a mixed strategy), where $B_i(b;\theta_i)$ is the cdf from which bidder i draws a bid of b when his type is θ_i. Let β_i be the density function associated with B_i.

Lemma 1 *Suppose there exist* $\theta_i \in \left[\underline{\theta}, \bar{\theta}\right]$ *and some* b *in the support of* β_i

level of matching. As a result, bidders can credibly signal their better matches by offering higher bids.

The optimal mechanism requires that if a contract is awarded, it is awarded to the bidder with the best match. Lemma 2 implies that this condition is satisfied for any symmetric separating equilibrium. Therefore, we will proceed by solving for the symmetric separating equilibria of the first-score auction game.

Lemma 3 *Let B be a symmetric separating equilibrium bidding strategy. If $\theta \in \left[\underline{\theta}, \bar{\theta}\right]$ and there exists some b in the support of $\beta(\cdot;\theta)$ such that $P(b$ (*

Lemma 5 *In any symmetric separating equlibrium, the bidding strategy is*

$$b(\theta) = \theta - \frac{\int_{\underline{\theta}}^{\theta} F^{n-1}(x)dx}{F^{n-1}(\theta)}$$

for $\theta \in (\underline{\theta}; \overline{\theta}]$.

Proof: See Appendix. $\square$

An interesting feature of the bidding function specified by Lemma 5 is that the seller's

Suppose $V(\underline{\mu}) + \underline{\mu} < 0$. In this case, $V(\mu_0) + \mu_0 = 0$ and $V(\mu_{¤}) + \mu_{¤} > 0$. Since V

commitment power. This may explain why we observe agents, such as PDVSA, who are able to commit to a di®erent allocation mechanism, simply implementing a ¯rst-score auction.

In the next section, we consider the value of information about the matches and ask whether the seller can do better if he observes the matches ex-ante.

4 The Value of Information

Suppose the seller could observe the vector of matches before selecting a mechanism for allocating the contract. Would it be in the seller's interest to do so? How does the seller's value for the information vary with his ability to commit to an allocation rule?

Under commitment, the seller can clearly do better by observing the matches in advance. With the information in hand, he simply makes a take-it-or-leave-it o®er to the bidder with the best match in the amount of that bidder's valuation. Since the outcome is e±cient and the seller extracts all the surplus, he necessarily improves upon the mechanism outlined in Proposition 1. Therefore, a seller with commitment power has a positive value for the information.

Suppose the seller cannot commit. In this case, the seller will not be able to capture the entire surplus. At best, the seller can extract the value-match combination, $V(\mu) + \mu$, of the bidder with the second-highest match since any package of greater value o®ered by the best-matched bidder goes uncontested by the other bidders. This suggests that the seller's value for the information is lower under the no commitment paradigm than under the commitment paradigm.

Our approach will be to revisit the first-score auction outlined in Section 3.2 under the assumption that the seller observes the matches in advance and compare equilibrium bids across the two information structures. In doing so, we will identify how the seller's ex ante knowledge of the matches alters the bidders' incentives.

In the first-score auction game with an informed seller, bidder i selects an equilibrium bidding strategy $B_i : \mathbb{R} \times [\underline{\theta}; \bar{\theta}] \to [0; 1]$ such that for any type $\theta_i \in [\theta$

allocates the contract to the bidder offering the highest score, provided that score is not less than zero, $Q_i(s)$ is simply the probability that s is the highest score offered if $s \geq 0$ and zero otherwise. Finally, let $\Sigma_i : R \times [\underline{\theta}; \overline{\theta}] \to [0; 1]$ represent bidder i's equilibrium score strategy (possibly a mixed strategy), where $\Sigma_i(s; \theta_i)$ is the cdf from which bidder i draws a score of s when his type is θ_i, and let σ_i be the density function associated with Σ_i.

Using this notation, we can now reformulate the bidder's problem. Bidder i selects an equilibrium score strategy Σ_i such that for any type $\theta_i \in \left[\underline{\theta}; \overline{\theta}\right]$ and any score s_i in the support of $\sigma_i(\cdot; \theta_i$

Lemma 6 indicates that for all $\mu > \mu_0$, higher types o®er higher scores. However, this

The asymmetry effect: A well matched bidder is preferred by the seller, and therefore, he need not bid as aggressively to win.[14]

The value effect induces bids to increase with match. It is the reason we observe monotonicity in bids in both the standard first-price auction and the first-score auction with an uninformed seller. It follows that the value effect is represented by the first two terms of the bidding function outlined in Proposition 3.

The asymmetry effect is represented by the third term. Since the seller's utility increases with match and the matches are known to the seller, well matched bidders have an advantage over poorly matched bidders: a bidder with a high match can bid less than a bidder with a low match and still win the contract. In other words, the asymmetry across bidders tends to dampen price competition. The greater the importance of matching, V', the greater the asymmetry and the lower the bids.

If the lowest participating type is fixed across information structures (i.e., if $\mu_* = \mu_0 = \underline{\mu}$), then bids are lower when the seller observes the matches in advance. In this case, the seller is clearly better off when he remains uninformed. But what if the lowest participating type differs across the two auctions (i.e., if $\mu_* > \mu_0$)?[15] In this case, the seller is still better off remaining uninformed since μ_* is the optimal reserve price. This observation delivers the following result:

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[14] The asymmetry effect is similar to "the competition effect" in Rezende (2004).

[15] When the matches are observed, the lowest participating type must be θ_0 since the seller is unable to commit to excluding bidders with higher types and there is no mutually beneficial trade between the seller and a bidder whose type is less than θ_0. However, when the matches are not observed in advance, the seller may be able to exclude bidders with types greater than θ_0 if he believes their types are sufficiently low.

Corollary 1 *In a first-score auction, the seller can raise his expected utility by choosing not to observe the matches in advance.*

Therefore, the seller's value for information about the matches is not only lower in the absence of commitment but may actually be negative.

We conclude this section by offering a more intuitive interpretation of the first-score auction with an informed seller. Without commitment, the seller is not able to reject an offer which exceeds his reservation value but cannot be matched by any other bidder. That is, if Bertrand competition drives offers up to s, the seller requires commitment power to reject any offer which exceeds s. Therefore, the best the seller can do is allocate the contract to the bidder with the best match, who, in turn, delivers a score equal to $V(\mu_2) + \mu_2$, where μ_2 is the second-highest match. The selling price is given by

$$p = \mu_2 - [V(\mu_1) - V(\mu_2)]; \tag{4.6}$$

where μ_1 is the highest match.[16]

This is precisely the outcome in a *second-score* auction. In a second-score auction, the

The second-score formulation yields a convenient interpretation of the asymmetry e®ect. In equation (4.6), the selling price p is less than the price o®ered by the bidder with the second-highest match, μ_2. The reduction is driven by the di®erence in the seller's values for the two matches, $V(\mu_1) - V(\mu_2)$. This term represents the advantage enjoyed by the best-matched bidder, or rather, the asymmetry e®ect. Once again, we see that the greater the importance of matching, V', the greater the asymmetry and the lower the selling price.

5 Conclusion

For a wide range of commercial arrangements, a good match between the buyer and seller raises the value of the contract for both parties. However, at the time the terms of the contract are set, the parties may not be fully informed about the degree to which they match. In this paper, we have addressed the case in which the quality of the match is the private information of the bidder.

The paper opened by asking whether the seller should account for matching in his allocation decision. It is shown that no matter how important matching is to the seller, he need not consider matching as a factor in order to implement the optimal mechanism. Since the bidder's value for the contract increases with match, allocating the contract to the highest bidder is equivalent to selecting the bidder with the best match.

However, it is also shown that if the seller cannot commit to allocating the contract on the basis of price alone, he can still implement the optimal mechanism. Since well matched bidders have a higher value for the contract, higher bids signal higher matches; as a result, the seller ¯nds that he has no incentive to contract with anyone other than the highest bidder. Moreover, if matching is sufficiently important, the seller can credibly commit to the prescribed reserve price by associating bids which fall below the reserve with a poor match.

This paper has also provided answers to the natural questions regarding the effects of information asymmetry, namely (1) What is the seller's value for the bidder's information? and (2) How would the equilibrium behavior of the bidders change if the seller were to observe the matches in advance? The answers to these questions hinge on whether the seller can credibly commit to an allocation rule ex ante which he would prefer to violate ex post.

We find that if the seller can commit, observing the matches in advance allows him to appropriate all the rent. Therefore, the value of the information is positive. However, if the seller cannot commit, observing the matches introduces an asymmetry across bidders that depresses bids. Consequently, the seller's value for the information may be negative.

This result is surprising for two reasons. First, we observe that the bidders capture less rent when their information is private. This follows from the fact that the seller's knowledge of the matches eliminates the need for well matched bidders to signal their favorable matches through higher bids. Well matched bidders can, instead, capitalize on their preferred status, bid less than their poorly matched counterparts, and still win the auction. Second, we observe that the more the seller cares about matching, the less it pays for him to observe the matches in advance. The intuition is that the more matching matters, the greater the advantage enjoyed by a well-matched bidder, and the larger the margin by which he can reduce his bid and still win.

with the mechanism outlined in Proposition 1, we demonstrate that the mechanism satisfies the constraints of the original program.

Suppose the mechanism $\{p(\cdot), t(\cdot)\}$ satisfies the IC and IR constraints for all $\theta_i \in [\underline{\theta}, \bar{\theta}]$ but that $U_i(\underline{\theta}, \underline{\theta}) = \varepsilon > 0$. Now consider a different mechanism $\langle p(\cdot), \hat{t}(\cdot) \rangle$, where $\hat{t}_i(\cdot) \equiv t_i(\cdot) + \varepsilon$. The mechanism $\langle p(\cdot), \hat{t}(\cdot) \rangle$ satisfies the IR constraint for all $\theta_i \in [\underline{\theta}, \bar{\theta}]$ since

$$\begin{aligned} \hat{U}_i(\theta_i, \theta_i) &= E_{\theta_{-i}}\left[\theta_i p_i(\theta_i, \theta_{-i}) - \hat{t}_i(\theta_i, \theta_{-i})\right] \\ &= E_{\theta_{-i}}\left[\theta_i p_i(\theta_i, \theta_{-i}) - t_i(\theta_i, \theta_{-i})\right] - \varepsilon \\ &\geq E_{\theta_{-i}}\left[\theta_i p_i(\underline{\theta}, \theta_{-i}) - t_i(\underline{\theta}, \theta_{-i})\right] - \varepsilon \\ &\geq E_{\theta_{-i}}\left[\underline{\theta} p_i(\underline{\theta}, \theta_{-i}) - t_i(\underline{\theta}, \theta_{-i})\right] - \varepsilon \\ &= 0. \end{aligned} \tag{A.1}$$

The first inequality holds because $\{p(\cdot), t(\cdot)\}$ satisfies the IC constraint; the second inequality holds because $p_i \in [0, 1]$. The mechanism $\langle p(\cdot), \hat{t}(\cdot) \rangle$ also satisfies the IC constraint for all $\theta_i \in [\underline{\theta}, \bar{\theta}]$ since

$$\begin{aligned} \hat{U}_i(\theta_i, \theta_i) &= E_{\theta_{-i}}\left[\theta_i p_i(\theta_i, \theta_{-i}) - t_i(\theta_i, \theta_{-i})\right] - \varepsilon \\ &\geq E_{\theta_{-i}}\left[\theta_i p_i(x, \theta_{-i}) - t_i(x, \theta_{-i})\right] - \varepsilon \\ &= \hat{U}_i(x, \theta_i) \end{aligned} \tag{A.2}$$

for all $x \in [\underline{\theta}, \bar{\theta}]$. The inequality follows from the fact that $\{p(\cdot), t(\cdot)\}$ satisfies the IC constraint. Since $\varepsilon > 0$ and $\hat{t}_i(\cdot) \equiv t_i(\cdot) + \varepsilon$, the seller's expected utility is strictly greater under $\langle p(\cdot), \hat{t}(\cdot) \rangle$ than it is under $\{p(\cdot), t(\cdot)\}$. Therefore, the original mechanism $\{p(\cdot), t(\cdot)\}$ cannot be optimal – a contradiction.

Having established that $U_i(\underline{\theta}, \underline{\theta}) = 0$, we turn our attention to developing a relaxed optimization program. Our approach is to modify the original program so as to reduce the number of choice variables from two, $p(\cdot)$ and $t(\cdot)$, to one, $p(\cdot)$. From equation (3.1), we have

$$E_{\theta_{-i}}\left[t_i(\theta_i, \theta_{-i})\right] = E_{\theta_{-i}}\left[\theta_i p_i(\theta_i, \theta_{-i})\right] - U_i(\theta_i, \theta_i), \tag{A.3}$$

and after substituting for $E_{\mu_{-i}}[t_i(\mu_i;\mu_{-i})]$ in equation (3.2), we obtain

$$U_0 = E_\mu\left[\sum_{i=1}^{n}(V(\mu_i)+\mu_i)\,p_i(\mu)\right] - \sum_{i=1}^{n} E_{\mu_i}[U_i(\mu_i;\mu_i)]. \tag{A.4}$$

It remains to develop an expression for $E_{\mu_i}[U_i(\mu_i;\mu_i)]$ in terms of $p_i(\mu)$.

Incentive compatibility requires that $U_i(\mu_i;\mu_i) \geq U_i(x;\mu_i)$ for all $\mu_i \in [\underline{\mu};\bar{\mu}]$

Using equation (A.10), we substitute for $E_{\mu_i} [U_i (\mu_i; \mu_i$

for all $\theta_i \in [\underline{\theta}, \bar{\theta}]$ and all $x \in [\underline{\theta}, \bar{\theta}]$. Subtracting $U_i(x;x)$ from both sides and applying equation (3.1) yields

$$U_i(\theta_i;\theta_i) - U_i(x;x) \ge (\theta_i - x)\, E_{\theta_{-i}}[p_i(x;\theta_{-i})]. \qquad \text{(A.13)}$$

If θ_i and x are less than θ_*, the inequality is trivially satisfied. If $\theta_i < \theta_*$ and $x \ge \theta_*$, the left-hand side is

$$\begin{aligned} U_i(\theta_i;\theta_i) - U_i(x;x) &= -\int_{\theta_*}^{x} F^{n-1}(y)dy \\ &\ge -(x - \theta_*) F^{n-1}(x) \\ &\ge -(x - \theta_i) F^{n-1}(x) \\ &= (\theta_i - x)\, E_{\theta_{-i}}[p_i(x;\theta_{-i})] \end{aligned} \qquad \text{(A.14)}$$

and the inequality is again satisfied. If $\theta_i \ge \theta_*$ and $x < \theta_*$, the inequality reduces to $\int_{\theta_*}^{\theta_i} F^{n-1}(y)dy \ge 0$, which is trivially satisfied. If $\theta_i \ge x \ge \theta_*$,

$$\begin{aligned} U_i(\theta_i;\theta_i) - U_i(x;x) &= \int_{x}^{\theta_i} F^{n-1}(y)dy \\ &\ge (\theta_i - x) F^{n-1}(x) \\ &= (\theta_i - x)\, E_{\theta_{-i}}[p_i(x;\theta_{-i})] \end{aligned} \qquad \text{(A.15)}$$

An analogous argument can be used to verify that the inequality holds if $x \ge \theta_i \ge \theta_*$. $\square$

Proof of Lemma 1: Suppose $\hat{b}$ is in the support of $\beta_i(\cdot;\hat{\theta}_i)$ and b is in the support of $\beta_i(\cdot;\theta_i)$. If $\hat{\theta}_i > \theta_i$ and $P_i(b) > P_i(\hat{b})$

$\hat{\theta}_i) b\ (x;)$

By definition of equilibrium,

$$(\mu_i - b)P_i(b) \geq (\mu_i - \hat{b})P_i(\hat{b}) \tag{A.17}$$

and

$$(\hat{\mu}_i - \hat{b})P_i(\hat{b}) \geq (\hat{\mu}_i - b)P_i(b). \tag{A.18}$$

Combining inequalities (A.17) and (A.18),

$$\mu_i\left[P_i(b) - P_i(\hat{b})\right] \geq bP_i(b) - \hat{b}P_i(\hat{b}) \geq \hat{\mu}_i\left[P_i(b) - P_i(\hat{b})\right]$$

By combining these two results, we find that a bidder with type $\theta \in [\underline{\theta}, \bar{\theta}]$ wins the auction if the type of every other bidder is strictly less than θ and loses the auction if there exists a bidder whose type is strictly greater than θ; that is, a bidder with type $\theta \in [\underline{\theta}, \bar{\theta}]$ wins the auction with probability $F^{n-1}(\theta)$.

Suppose a bidder with type θ plays the bidding strategy $B(\cdot;\theta)$ in equilibrium. Suppose further that $B(\cdot;\theta)$ is a mixed strategy and that the support of $\beta(\cdot;\theta)$ includes the bids b and b', where $b \neq b'$. Since the bidder should be indifferent among bids in the support of $\beta(\cdot;\theta)$,

$$(\theta - b)F^{n-1}(\theta) = (\theta - b')F^{n-1}(\theta). \tag{A.21}$$

Since $\theta \geq \underline{\theta}$, $F^{n-1}(\theta)$ must be positive, which in turn implies that b and b' are equal – a contradiction. Hence, the bidding strategy $B(\cdot;\theta)$ must be a pure strategy.

An analogous argument can be used to prove the lemma for the case in which a bidder with type $\underline{\theta}$ wins the auction with probability zero. $\square$

Proof of Lemma 4: Suppose $b : (\underline{\theta}, \bar{\theta}] \to \mathbb{R}$ is not continuous at some $\theta \in (\underline{\theta}, \bar{\theta}]$. Then, for some $\epsilon > 0$, there is no $\delta > 0$ such that

$$\hat{\theta} \in (\underline{\theta}, \bar{\theta}] \text{ and } \left|\hat{\theta} - \theta\right| < \delta \Rightarrow \left|b(\hat{\theta}) - b(\theta)\right| < \epsilon. \tag{A.22}$$

By Lemma 2, b is strictly increasing on $(\underline{\theta}, \bar{\theta}]$. Therefore, we can restate the discontinuity condition as follows: there exists $\epsilon > 0$ such that either

$$b(\theta) - b(\hat{\theta}) \geq \epsilon, \quad \forall \hat{\theta} \in (\underline{\theta}, \theta) \tag{A.23}$$

or

$$b(\hat{\theta}) - b(\theta) \geq \epsilon, \quad \forall \hat{\theta} \in \left(\theta, \bar{\theta}\right]. \tag{A.24}$$

Incentive compatibility for the type θ bidder requires that

$$[\theta - b(\theta)]F^{n-1}(\theta) \geq [\theta - b(\hat{\theta})]F^{n-1}(\hat{\theta}) \tag{A.25}$$

for all $\hat{\theta} \in (\tilde{\theta}, \bar{\theta}]$. If condition (A.23) holds, then

$$\left[\theta - b(\hat{\theta})\right]\left[F^{n-1}(\theta) - F^{n-1}(\hat{\theta})\right] \geq \epsilon F^{n-1}(\theta) \tag{A.26}$$

for all $\hat{\theta} \in (\tilde{\theta}, \theta)$. Since $\theta \in (\hat{\theta}, \bar{\theta}]$ and $\epsilon > 0$ are fixed, $\epsilon F^{n-1}(\theta)$ is both positive and fixed. However, since F is continuous on $\left[\underline{\theta}, \bar{\theta}\right]$, F^{n-1} is also continuous, which implies that for all $\epsilon' > 0$, there exists $\delta' > 0$ such that

$$\hat{\theta} \in (\tilde{\theta}, \theta) \text{ and } \theta - \hat{\theta} < \delta' \Rightarrow F^{n-1}(\theta) - F^{n-1}(\hat{\theta}) < \epsilon';$$

that is, $F^{n-1}(\theta) - F^{n-1}(\hat{\theta})$ can be brought arbitrarily close to zero by selecting a $\hat{\theta}$ sufficiently close to θ. Moreover, Lemma 2 indicates that b is increasing over $(\tilde{\theta}, \bar{\theta}]$, and since θ is fixed, $\theta - b(\hat{\theta})$ is decreasing as $\hat{\theta}$ approaches θ. Therefore, inequality (A.26) is violated for $\hat{\theta}$ sufficiently close to θ, and condition (A.23) cannot hold.

An analogous argument using incentive compatibility for the type $\hat{\theta}$ bidder can be used to show that condition (A.24) cannot hold either. $\square$

Proof of Lemma 5: Our approach will be to first establish a boundary condition by showing that

$$\lim_{\theta \to \tilde{\theta}^{+}} b(\theta) = \tilde{\theta}$$

Consider a bidder with type $\theta \in (\underline{\theta}; \bar{\theta}]$. If the bidder offers a bid of $b(x)$, where $x \in (\underline{\theta}; \bar{\theta}]$, then the bidder's expected utility is

$$U(x; \theta) \equiv [\theta - b(x)] F^{n-1}(x). \tag{A.31}$$

In equilibrium, $b(\cdot)$ must satisfy

Global IC : $\quad U(\theta; \theta) \geq U(x; \theta); \quad \forall \theta \in (\underline{\theta}; \bar{\theta}]; \forall x \in (\underline{\theta}; \bar{\theta}]$.

Since b and F are continuous, Global IC implies that $b(\cdot)$ satisfies

Local IC : $\quad U_x(\theta; \theta) = 0 \quad \forall \theta \in (\underline{\theta}; \bar{\theta})$.

Taking the derivative of $U(x; \theta)$ with respect to x, substituting θ for x, and setting the resulting expression equal to zero yields

$$\frac{db(\theta)}{d\theta} F^{n-1}(\theta) + b(\theta) \frac{dF^{n-1}(\theta)}{d\theta} = \theta \frac{dF^{n-1}(\theta)}{d\theta}. \tag{A.32}$$

After integrating both sides, evaluating the integrals from $\underline{\theta}$ to θ, applying the boundary condition ($\lim_{\theta \to \underline{\theta}_+} b(\theta) = \underline{\theta}$), and solving for $b(\theta)$, we obtain the bidding function

$$b(\theta) = \theta - \frac{\int_{\underline{\theta}}^{\theta} F^{n-1}(x) dx}{F^{n-1}(\theta)} \tag{A.33}$$

for $\theta \in (\underline{\theta}; \bar{\theta})$. Since $b(\cdot)$ is continuous over $(\underline{\theta}; \bar{\theta}]$, the bidding function specified gives the equilibrium bid for type $\bar{\theta}$ as well. $\Box$

Proof of Proposition 2: Our approach will be to derive the equilibrium expected utility for a bidder with type $\theta \in \left[\underline{\theta}; \bar{\theta}\right]$ and then show that there is no profitable deviation available to that bidder.

Consider a bidder with type $\mu \in [\underline{\mu}, \mu_*)$. Since the equilibrium is separating, the seller can infer μ from $b(\mu)$, and the bidder's score is given by

$$\begin{aligned} V(\mu) + b(\mu) &< V(\mu_*) - V(\mu_*) \\ &= 0. \end{aligned} \tag{A.34}$$

Since the seller's reservation utility is zero, the bidder will not be awarded the contract. Hence, any bidder with type $\mu \in [\underline{\mu}, \mu_*)$ earns utility of zero in equilibrium.

Consider a bidder with type $\mu \in [\mu_*, \bar{\mu}]$. In this case, the bidder's score is given by

$$V(\mu) + \mu - \frac{\int_{\mu_*}^{\mu} F^{n-1}(x)dx}{F^{n-1}(\mu)}. \tag{A.35}$$

Since the score is strictly increasing in μ, the bidder's probability of winning the contract is $F^{n-1}(\mu)$. Hence, in equilibrium, any bidder with type $\mu \in [\mu_*, \bar{\mu}]$ earns expected utility of

$$[\mu - b(\mu)]\,F^{n-1}(\mu). \tag{A.36}$$

We now show that no bidder has an incentive to deviate to another bid on the equilibrium path. Suppose a bidder has type $\mu \in [\underline{\mu}, \bar{\mu}]$. If this bidder deviates to a bid $b(x)$, where $x \in [\underline{\mu}, \mu_*)$, then his resulting utility is zero, which does not improve upon his equilibrium expected utility. Similarly, if the bidder deviates to a bid $b(x)$, where $x \in [\mu_*, \bar{\mu}]$, then his expected utility is

$$[\mu - b(x)]\,F^{n-1}(x). \tag{A.37}$$

Substituting for $b(x)$ yields

$$(\mu - x)F^{n-1}(x) + \int_{\mu_*}^{x} F^{n-1}(y)dy, \tag{A.38}$$

which is increasing in x for $x < \mu$ and decreasing in x for $x > \mu$. Therefore, if $\mu \in [\mu_*, \bar{\mu}]$, bidding $b(x)$ delivers lower expected utility than bidding $b(\mu)$. Furthermore, if $\mu \in [\underline{\mu}, \mu_*)$, then μ is strictly less than $b(x)$, and $b(x)$ cannot be a profitable deviation.

Finally, we show that no bidder has an incentive to deviate to a bid off the equilibrium path. Consider the deviating bid $b > b(\bar{\mu})$. If the bidder's type, μ, is less than b, then the expected utility associated with b must be nonpositive, and b cannot be a profitable deviation. If, instead, $\mu \geq b$, then the expected utility associated with b is at best $\mu - b$, which is strictly less than the expected utility associated with bidding $b(\bar{\mu})$. Since $b(\bar{\mu})$ is not a profitable deviation for the bidder, then b is not a profitable deviation either.

Now consider the deviating bid $b < \mu_*$. The score associated with b is given by

$$\begin{aligned} V(\underline{\mu}) + b &< V(\underline{\mu}) + \mu_* \\ &= \frac{1 - F(\mu_*)}{f(\mu_*)} - [V(\mu_*) - V(\underline{\mu})] \qquad \text{(A.39)} \\ &\leq 0. \end{aligned}$$

Since the seller's reservation utility is zero, the contract is never awarded to a bidder offering a bid of b, and b cannot be a profitable deviation. $\Box$

References

[1] Asker, J. and E. Cantillon (2004), "Properties of Scoring Auctions," Discussion Paper 4734, C.E.P.R.

[2] Avery, C. (1998), "Strategic Jump Bidding in English Auctions," *Review of Economic Studies*, 65(2), 185-210.

[3] Benoît, J. -P. and J. Dubra (forthcoming), \Information Revelation in Auctions," *Games and Economic Behavior*.

[4] Bester, H. and R. Strausz (2000), \Imperfect Commitment and the Revelation Principle: The Multi-Agent Case," *Economics Letters*, 69(2), 165-171.

[5] Bikhchandani, S. and C. -F. Huang (1989), \Auctions with Resale Markets: An Exploratory Model of Treasury Bill Markets," *Review of Financial Studies*, 2(3), 311-339.

[6] Branco, F. (1997), \The Design of Multidimensional Auctions," *RAND Journal of Economics*, 28(1), 63-81.

[7] Bulow, J. (1982), \Durable-Goods Monopolists," *Journal of Political Economy*, 90(2), 314-332.

[8] Bulow, J. and P. Klemperer (1996), \Auctions versus Negotiations," *American Economic Review*, 86(1), 180-194.

[9] Bulow, J. and J. Levin (forthcoming), \Matching and Price Competition," *American Economic Review*.

[10] Bulow, J. and J. Roberts (1989), \The Simple Economics of Optimal Auctions," *Journal of Political Economy*, 97(5), 1060-1090.

[11] Burguet, R. and J. Sákovics (1996), \Reserve Prices without Commitment," *Games and Economic Behavior*, 15(2), 149-164.

[12] Cantillon, E. (2005), \The E®ect of Bidders' Asymmetries on Expected Revenue in Auctions," mimeo.

[13] Chalot, J. P. (1996, October 31), Personal Interview, Petr¶leos de Venezuela.

[14] Che, Y. -K. (1993), \Design Competition through Multidimensional Auctions," *RAND Journal of Economics*, 24(4), 668-680.

[15] Cho, I. K. and D. M. Kreps (1987), \Signaling Games and Stable Equilibria," *Quarterly Journal of Economics*, 102(2), 179-221.

[16] Coase, R. H. (1972), \Durability and Monopoly," *Journal of Law and Economics*, 15(1), 143-149.

[17] Crawford, V. P. and E. M. Knoer (1981), \Job Matching with Heterogeneous Firms and Workers," *Econometrica*, 49(2), 437-450.

[18] Dalkir, S., J. Logan, and R. Mason (2000), \Mergers in Symmetric and Asymmetric Noncooperative Auction Markets: The E®ects of Prices and E±ciency," *International Journal of Industrial Organization*, 18(3), 383-413.

[19] Das Varma, G. (2003), \Bidding for a Process Innovation under Alternative Modes of Competition," *International Journal of Industrial Organization*, 21(1), 15-37.

[20] Gale, D. and L. S. Shapley (1962), \C303(Sl8Se97((1Admisnal)s97(and)-297(L.)the97((1ble)t)26(y

[39] McAfee, R. P. and J. McMillan (1992), \Bidding Rings," *American Economic Review*, 82(3), 579-599.

[40] McAfee, R. P. and D. Vincent (1997), \Sequentially Optimal Auctions," *Games and Economic Behavior*, 18(2), 246-276.

[41]

[49] Shapley, L. S. and M. Shubik (1972), \The Assignment Game I: The Core," *International Journal of Game Theory*, 1(1), 111-130.

[50] Skreta, V. (2004), \Optimal Auction Design under Non-Commitment," mimeo.

[51] Stokey, N. (1981), \Rational Expectations and Durable Goods Pricing," *Bell Journal of Economics*, 12(1), 112-128.

[52] Tschantz, S., P. Crooke, and L. Froeb (1997), \Mergers in First versus Second Price Asymmetric Auctions," mimeo.

[53] Vartiainen, H. (2005), \Auction Design without Commitment," mimeo.

[54] Waehrer, K. (1999), \Asymmetric Private Values Auctions with Application to Joint Bidding and Mergers," *International Journal of Industrial Organization*, 17(3), 437-452.

[55] Waehrer, K. and M. K. Perry (2003), \The E®ects of Mergers in Open-Auction Markets," *RAND Journal of Economics*, 34(2), 287-304.

[56] Zheng, C. Z. (2000), \Optimal Auction in a Multidimensional World," Discussion Paper 1282, Northwestern University, Center for Mathematical Studies in Economics and Management Science.